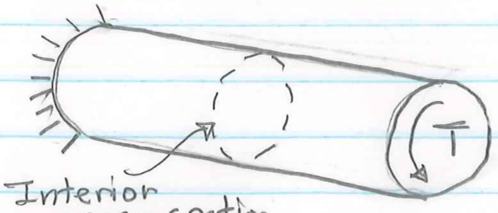
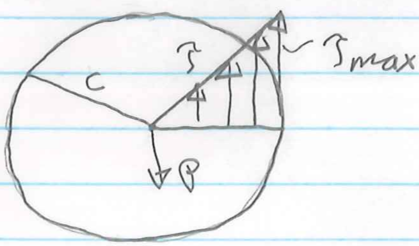
10. Twisting of a circular shaft
(linearly elastic, homogeneous, isotropic material)



T is the resultant torque couple about the axis of the shaft of the shear stresses applied to the end of the shaft. The resultant force associated with these shear stresses is zero.

Examine the shear stresses on a cross-section sufficiently far from the ends. This will be a St. Venant solution and the stresses take a very simple form: circumferential and varying linearly with the radial coordinate ρ.



Interior cross-section

$$\tau = \tau_{max}\frac{\rho}{c}$$

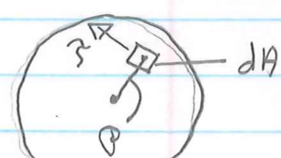
also,

$$\gamma = \frac{\tau}{G} = \frac{\tau_{max}}{G}\frac{\rho}{c} = \gamma_{max}\frac{\rho}{c}$$

From equilibrium, the resultant torque on every cross-section is T.

$$T = \int_{C-S} \rho\,\tau\,dA = \frac{\tau_{max}}{c}\int_{C-S} \rho^2\,dA$$

$$= \frac{\tau_{max}}{c}J$$



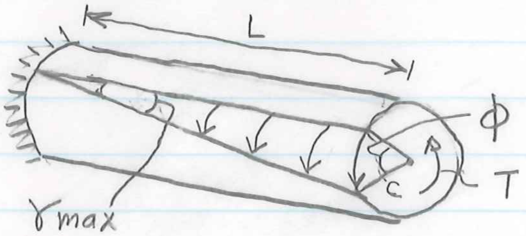
where

$$J = \int_{C-S} \rho^2\,dA = 2\pi\int_0^c \rho^3\,d\rho = \frac{1}{2}\pi c^4$$

J is the polar moment of inertia of the cross-section.

Also: $\tau_{max} = \dfrac{Tc}{J}$ and $\tau = \dfrac{T\rho}{J}$

The linear distribution of γ with ρ means that each circular cross-section rotates as a rigid planar section. If we assume that our St. Venant solution holds along the entire length of the shaft, the displaced configuration appears as below.



From geometry, $\gamma_{max} = \frac{\phi c}{L}$ where ϕ is the rotation of the end of the shaft

At some interior point, $\gamma = \frac{\phi \rho}{L}$

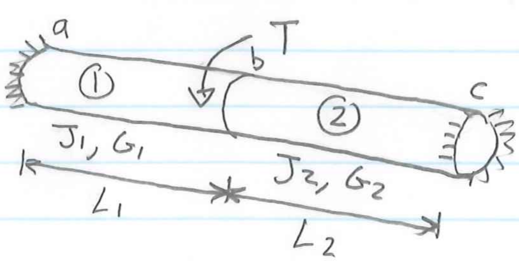
The relation between T and ϕ can be derived as follows.

$$\gamma_{max} = \tau_{max} / G$$

$$\frac{\phi c}{L} = \frac{Tc}{JG}$$

$$T = \frac{JG}{L}\phi$$ where $\frac{JG}{L}$ is the rotational stiffness of the shaft in torsion. (Compare to $\frac{AE}{L}$ for axial stiffness.)

Example:

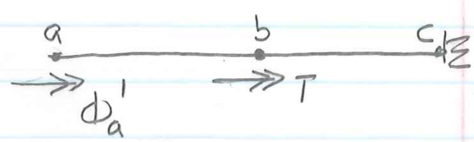


Bar consists of two pieces and is fixed at both ends. T is applied where the two pieces join.

Find the reaction couple Ra at the left end. Use superposition and make statically determinant by removing the left support.

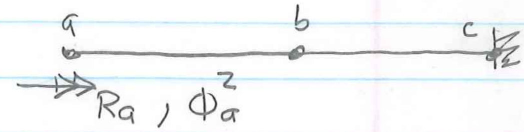
Step 1. Apply T

$$\phi_a^1 = T / \frac{J_2 G_2}{L_2}$$



Step 2. Apply R_a

$$\phi_a^2 = R_a\left[\frac{1}{J_1G_1/L_1} + \frac{1}{J_2G_2/L_2}\right]$$

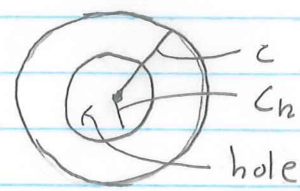


Step 3. $\phi_a^1 + \phi_a^2 = 0$

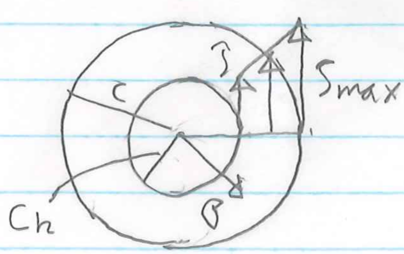
Substitute and solve for R_a: $R_a = -T\dfrac{J_1G_1/L_1}{J_1G_1/L_1 + J_2G_2/L_2}$

(Compare this to the result on page 15 of the notes)

What if the shaft is hollow?



In the St. Venant solution, the shear stress τ is still circumferential and still a linear function of ρ.



$$\tau = \tau_{max}\frac{\rho}{c} \quad \text{for } \rho \geq c_h$$

All of the equations are the same as presented previously, except J is given as

$$J = \int_{c-s} \rho^2 dA = 2\pi\int_{c_h}^{c} \rho^3 d\rho = \tfrac{1}{2}\pi\left(c^4 - c_h^4\right)$$